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Classification of the BPS states in Bagger-Lambert theory

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Abstract: We classify, in a group theoretical manner, the BPS configurations in the multiple M2-brane theory recently proposed by Bagger and Lambert. We present three types of BPS equations preserving various fractions of supersymmetries: in the first type we have constant fields and the interactions are purely algebraic in nature; in the second type the equations are invariant under spatial rotation SO(2), and the fields can be time-dependent; in the third class the equations are invariant under boost SO(1, 1) and provide the eleven-dimensional generalizations of the Nahm equations. The BPS equations for different number of supersymmetries exhibit the division algebra structures: octonion, quarternion or complex.

Keywords: Brane Dynamics in Gauge Theories, M-Theory.

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Contents

1. Introduction

In a series of recent papers [1], Bagger and Lambert (BL) have constructed a three-dimensional, interacting superconformal gauge theory of multiple M2-branes. The action is maximally supersymmetric with 16 ordinary supersymmetries, and it has been verified that the theory is indeed superconformal with 16 conformal supercharges in [2]. In the quest for the final form of the theory, as usual, it was supersymmetry that provided crucial guiding lights. The work was initiated as an attempt to incorporate Basu and Harvey's generalized Nahm equation -which was a proposal to describe M2-branes ending on an M5-brane [3]- in the full supersymmetric M2-brane action. Their analysis revealed a novel algebraic structure, namely the 3-algebra, which is also investigated independently by Gustavsson [4]. Since the discovery, the multiple M2-brane theory of Bagger and Lambert has attracted an enormous degree of attention [5 − 35]. One might expect that, given this genuine superconformal field theory, $\mathcal{M}$-theory is now about to unveil its mysterious and fundamental features.

In the present paper, we set out to classify the BPS states, or the BPS equations of the BL theory using a group theoretical consideration. Apparently the theory of our interest has the Lorentz group $\mathrm{SO}(1,2)$ and the R-symmetry group $\mathrm{SO}(8)$. Instead of providing the full and thorough survey of possible BPS equations, we focus mainly on two different types of BPS equations with different number of supersymmetries, and classify them completely. The first class is completely Lorentz invariant, and the other is invariant under the spatial rotation.

In the first type, the BPS equations are given purely in terms of the three-algebra commutators and independent of the three-dimensional worldvolume coordinates. Thus the corresponding nontrivial configurations possess infinite energy, typically corresponding to BPS objects of infinite size. Previously known analogous algebraic soultions include the longitudinal M5-brane in $\mathcal{M}$-theory matrix model which is realized in terms of Heisenberg algebra or large N matrices [36].

In the other type the equations are SO(2) rotation invariant, and the fields can be *time-dependent*. A technical reason why we focus on the two classes is that in these cases, fully utilizing the SO(8) triality we are able to classify the BPS equations completely.

In addition to the two classes, there is another possibility to obtain third type of BPS equations *via* simple tensor product. Namely one can obtain various generalizations of the Nahm equations which are invariant under the boost $\mathrm{SO}(1,1) \subset \mathrm{SO}(1,2)$. Our BPS equations manifest the division algebra structures: octonion, quarternion or complex. In the paper we will mainly focus on the BPS equations themselves. Our results hold for both the finite and infinite dimensional three-algebras. Note however that the Lorentz

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invariant BPS equations can have nontrivial solutions only for infinite dimensional three-algebras. The specific solutions and the physical interpretation will be presented in a separate publication [37].

The organization of the present paper is as follows. Section 2 is for preliminaries. We first discuss the general features of the 'supersymmetric projection matrices' and review how to derive the corresponding BPS equations for a given projection matrix. We also explain the relevant symmetries. Then we classify the projection matrices for the SO(1, 2), SO(2) and SO(1, 1) invariant equations. Section 3 contains our main results of the BPS equations. Section 3.1 classifies the SO(1, 2) invariant BPS equations preserving two, four, six, eight, ten and twelve supersymmetries.[1] Section 3.2 classifies the SO(2) invariant BPS equations preserving two, four, six and eight supersymmetries. In section 3.3 we discuss the SO(1, 1) invariant BPS equations which generalize the Nahm equations. The final section, section 4 contains our results and discussions. In appendix we review the SO(8) triality and its relation to octonions.

Note added. While this paper is being finished, ref. [38] appears in ArXiv which partially overlaps with our work, as it discusses the BPS equations of the form: $D_y X_I = \frac{1}{3!}\mathcal{C}_{IJKL}[X^J, X^K, X^L]$. In the present paper, we explicitly spell the coefficients $\mathcal{C}_{IJKL}$ and classify various BPS equations.

2. Preliminaries

The multiple M2-brane theory has 8 real scalar fields X^I, $I = 1, 2, \ldots, 8$ and a 16 component Majorana spinor Ψ. The supersymmetry transformation of the fermions in the Bagger-Lambert theory assumes the form:

$$\delta\Psi = \left(F_{\mu I}\Gamma^{\mu I} - \frac{1}{6}F_{IJK}\Gamma^{IJK}\right)\varepsilon\,, \tag{2.1}$$

where all the variables are three-algebra valued and we set

$$F_{\mu I} \equiv D_\mu X_I\,, \qquad F_{IJK} \equiv [X_I, X_J, X_K]\;. \tag{2.2}$$

The bracket $[X_I, X_J, X_K]$ denotes the three-algebra product which is trilinear and totally antisymmetric. Note also that in contrast to the original convention [1] we let $I = 1, 2, \ldots, 8$ and take $\mu \equiv 0, 9, 10$ directions as for the M2-brane worldvolume for convenience to present the BPS equations later,

$$x^0 \equiv t\,, \qquad x^9 \equiv x\,, \qquad x^{10} \equiv y\,. \tag{2.3}$$

The supersymmetry parameter is real and subject to the SO(1, 2) projection condition:

$$\Gamma^{txy}\varepsilon = \varepsilon\,, \tag{2.4}$$

[1] Note that in the present paper we focus on the sixteen ordinary supersymmetries and not the sixteen conformal supersymmetries. For the BPS equations preserving conformal supersymmetries in super Yang-Mills we refer the readers to ref. [39].

which is consistent with the opposite projection property, $\Gamma^{txy}\Psi = -\Psi$. Since the product of all the eleven-dimensional gamma matrices leads to the 32×32 identity matrix $\Gamma^{txy123\cdots 8} = 1$, the above SO(1,2) projection condition coincides with the chirality condition of SO(8),

$$\Gamma^{123\cdots 8}\varepsilon = \varepsilon\,. \tag{2.5}$$

2.1 Supersymmetry projection matrix - general

In general for supersymmetric theories, the supersymmetry projection matrix Ω can be defined in terms of the commuting, real, orthonormal supersymmetry parameters $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N$,

$$\Omega := \sum_{i=1}^{N} \varepsilon_i \varepsilon_i^\dagger\,, \qquad \varepsilon_i^\dagger \varepsilon_j = \delta_{ij}\,, \tag{2.6}$$

satisfying $\Omega^\dagger = \Omega^2 = \Omega$. Here N denotes the number of the preserved supersymmetries,

$$N = \mathrm{Tr}\Omega\,. \tag{2.7}$$

Naturally the eigenvalues of the projection matrices are either zero or one.

When the supersymmetry transformation of fermions takes the form $\delta\Psi = \mathcal{F}\varepsilon$ where $\mathcal{F}$ denotes a bosonic quantity contracted with gamma matrices as in (2.1), the general strategy to obtain the BPS equations is as follows [40]:

1. *Expand the projection matrix Ω in terms of the gamma matrix product basis.*

2. *Perform the matrix product $\mathcal{F}\Omega$ and reexpress it in terms of the gamma matrix product basis.*

3. *Read off the BPS equations from the coefficients of the linearly independent terms.*

For example in the Euclidean four-dimensional minimal super Yang-Mills theory, we have two choices for the projection matrix $\Omega = \frac{1}{2}(1 \pm \gamma^{1234})$, while $\mathcal{F} = F_{ij}\gamma^{ij}$. Consequently, noting $\gamma^{12}\Omega = \mp\gamma^{34}\Omega$ *etc.*, we get $F_{ij}\gamma^{ij}\Omega = 2(F \mp \star F)_{i4}\gamma^{i4}\Omega$ such that the corresponding BPS equations are the well-known self-dual or anti-self-dual equations $F = \pm\star F$. In this way, the complete classifications of the BPS equations in six and eight-dimensional super Yang-Mills as well as the pp-wave M-theory matrix model [41] have been carried out [40, 42, 43].

The present paper concerns the BPS equations of the Bagger-Lambert theory. Since the eleven-dimensional spacetime admits Majorana spinors we can set all the gamma matrices and the spinors to be real. In particular, the spatial gamma matrices are symmetric while the temporal gamma matrix is anti-symmetric. Consequently, also from (2.5), the projection matrices of the Bagger-Lambert theory must satisfy

$$\Omega = \Omega^T = \Omega^*\,, \qquad \Omega = \Omega^2\,, \qquad \Omega = \mathcal{P}\Omega = \Omega\mathcal{P}\,, \tag{2.8}$$

where $\mathcal{P}$ is the SO(8) chiral projection matrix,

$$\mathcal{P} := \frac{1}{2}(1 + \Gamma^{123\cdots 8})\,. \tag{2.9}$$

The most general form of such projection matrices reads

$$\Omega = \left[c + \Upsilon_4 + \Gamma^x(c' + \Upsilon'_4) + \Gamma^y(c'' + \Upsilon''_4) + \Gamma^{xy}\Upsilon_2\right]\mathcal{P}\,, \tag{2.10}$$

where c, c', c'' are constants, $\Upsilon_4, \Upsilon'_4, \Upsilon''_4$ are foursome productions of the SO(8) gamma matrices Γ^{IJKL} contracted with self-dual four-forms, and Υ_2 is a twosome production of the SO(8) gamma matrices Γ^{IJ} contracted with a two-form. All together, *a priori*, there are $3 + 3 \times \frac{1}{2}\binom{8}{4} + \binom{8}{2} = 136$ real parameters which must be determined by requiring the remaining condition $\Omega^2 = \Omega$. The symmetry group $\mathrm{SO}(1,2)\times\mathrm{SO}(8)$ in the Bagger-Lambert theory may reduce the number of the free parameters, but is not big enough to transform all the free parameters, the two-form and the four-forms, into 'canonical' forms. Note that the SO(8) rotation may take only one of $\{\Upsilon_4, \Upsilon'_4, \Upsilon''_4, \Upsilon_2\}$ into a canonical form. In our choice, the canonical form of a two-form reads

$$\Upsilon_2 = a_1\Gamma^{12} + a_2\Gamma^{34} + a_3\Gamma^{56} + a_4\Gamma^{78}\,, \tag{2.11}$$

while the canonical form of a self-dual four-form reads

$$\Upsilon_4 = b_1\mathcal{E}_1 + b_2\mathcal{E}_2 + b_3\mathcal{E}_3 + b_4\mathcal{E}_4 + b_5\mathcal{E}_5 + b_6\mathcal{E}_6 + b_7\mathcal{E}_7\,, \tag{2.12}$$

where we set

$$\begin{array}{llll} \mathcal{E}_1 = \Gamma_{8127}\mathcal{P}\,, & \mathcal{E}_2 = \Gamma_{8163}\mathcal{P}\,, & \mathcal{E}_3 = \Gamma_{8246}\mathcal{P}\,, & \mathcal{E}_4 = \Gamma_{8347}\mathcal{P}\,, \\ \mathcal{E}_5 = \Gamma_{8567}\mathcal{P}\,, & \mathcal{E}_6 = \Gamma_{8253}\mathcal{P}\,, & \mathcal{E}_7 = \Gamma_{8154}\mathcal{P}\,. & \end{array} \tag{2.13}$$

The former is well known, while the latter is less familiar and we review it in appendix A. In (2.13) the subscript spatial indices of the gamma matrices are organized such that the three indices after the common 8 are identical to those of the totally anti-symmetric octonionic structure constants [40, 44]:

$$\begin{array}{ll} e_ie_j = -\delta_{ij} + c_{ijk}\,e_k\,, & i,j,k = 1,2,\ldots,7\,, \\ 1 = c_{127} = c_{163} = c_{246} = c_{347} = c_{567} = c_{253} = c_{154}\,, & \text{others zero}\,. \end{array} \tag{2.14}$$

We say Ω is invariant under SO(2) rotation invariant on xy-plane if $[\Gamma_{xy}\,,\,\Omega] = 0$. When this holds, for a finite angle ϕ and rotation $G = e^{\phi\Gamma_{xy}}$, from the equivalence

$$\mathcal{F}\Omega = 0 \qquad \Longleftrightarrow \qquad G\mathcal{F}\Omega G^{-1} = G\mathcal{F}G^{-1}\Omega = 0\,, \tag{2.15}$$

we note that the corresponding BPS equations are, as a set, invariant under the rotation. Naturally this generalizes to an arbitrary subgroup of $\mathrm{SO}(1,2)\times\mathrm{SO}(8)$.

In the present paper instead of attempting to solve for the most general projection matrices, we restrict to the cases where Ω assumes the canonical form. Namely we focus on two types of the BPS equations and classify the corresponding BPS equations completely: one is the $\mathrm{SO}(1,2)$ *invariant cases* i.e.

$$\Omega = (c + \Upsilon_4)\,\mathcal{P}\,, \tag{2.16}$$

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and the other is the $\mathrm{SO}(2)^{\mathbf{5}} \equiv \mathrm{SO}(2)\times \mathrm{SO}(2)\times \mathrm{SO}(2)\times \mathrm{SO}(2)\times \mathrm{SO}(2)$ *invariant cases* i.e.

$$\Omega = \big(\text{constant} + \text{twosome products of } \left\{\Gamma^{xy},\Gamma^{12},\Gamma^{34},\Gamma^{56},\Gamma^{78}\right\}\big)\,\mathcal{P}\,. \qquad (2.17)$$

Here SO(1,2) and SO(2) correspond to the M2 worldvolume Lorenz symmetry and the Cartan subgroup of the symmetry group $\mathrm{SO}(1,2)\times\mathrm{SO}(8)$ respectively. In addition, the former will easily generate various *$\mathcal{M}$-theoretic generalizations of the Nahm equations* which are invariant under $\mathrm{SO}(1,1)\subset\mathrm{SO}(1,2)$, as the corresponding projection matrices are of the form:

$$\Omega = (1\pm\Gamma^{tx})\,(c+\Upsilon_4)\,\mathcal{P}\,. \qquad (2.18)$$

2.2 SO(1,2) invariant projection matrices

The basic building blocks of all the possible SO(1,2) invariant projection matrices are the following $N=2$ projection matrices [40]:

$$\Omega = \frac{1}{8}\left(\mathcal{P} + \alpha_1\alpha_2\mathcal{E}_1 + \alpha_1\alpha_3\mathcal{E}_2 + \alpha_3\mathcal{E}_3 + \alpha_2\mathcal{E}_4 + \alpha_1\mathcal{E}_5 + \alpha_1\alpha_2\alpha_3\mathcal{E}_6 + \alpha_2\alpha_3\mathcal{E}_7\right)\,, \qquad (2.19)$$

where α_1, α_2, α_3 are three independent signs,

$$\alpha_1^2=\alpha_2^2=\alpha_3^2=1\,. \qquad (2.20)$$

Three independent sign choices lead to eight possible combinations, hence eight $N=2$ projection matrices. They are orthogonal to each other and complete, as summing all of them gives an identity. Namely they form an orthogonal basis for the SO(1,2) invariant projection matrices. General $N=2k$ projection matrices can be straightforwardly obtained as a k sum of the above eight $N=2$ projection matrices. Furthermore, from the SO(8) triality, the $8!/[k!(8{-}k)!]$ possibilities for the k sum are all equivalent to each other. The corresponding $N{=}\,2k$ BPS equations are $\mathrm{SO}(1,2)\times\mathrm{SO}(8{-}k)\times\mathrm{SO}(k)$ invariant.

2.3 SO(2) invariant projection matrices

The basic building blocks of all the possible SO(2) invariant projection matrices are the following $N=2$ projection matrices (see appendix B for derivation):

$$\begin{aligned}\Omega &= \frac{1}{8}\left[1{+}\Gamma^{xy}\big(\beta_1\Gamma^{12}{+}\beta_2\Gamma^{34}{+}\beta_3\Gamma^{56}{+}\beta_1\beta_2\beta_3\Gamma^{78}\big){-}\beta_1\beta_2\Gamma^{1234}{-}\beta_3\beta_1\Gamma^{1256}{-}\beta_2\beta_3\Gamma^{1278}\right]\mathcal{P}\\ &= \frac{1}{8}(1{+}\beta_1\Gamma^{xy12})(1{+}\beta_2\Gamma^{xy34})(1{+}\beta_3\Gamma^{xy56})\mathcal{P}\,, \qquad (2.21)\end{aligned}$$

where β_1, β_2, β_3 denote three independent signs,

$$\beta_1^2=\beta_2^2=\beta_3^2=1\,. \qquad (2.22)$$

Eight possible $N=2$ projection matrices form an orthogonal basis for the SO(2) invariant projection matrices. General $N=2k$ projection matrices can be straightforwardly obtained as a k sum of the above eight $N=2$ projection matrices. However, if the sum contains a pair of two opposite overall sign factors e.g. $(+++)$ and $(---)$, the corresponding BPS configurations become SO(1,2) invariant as $F_{\mu I}=0$ and the BPS equations reduce to those of SO(1,2) invariant BPS equations. Excluding these cases, up to SO(8) rotations, there are five inequivalent SO(2) invariant projection matrices as follows.

- $N=2$ SO(2)× SU(4) invariant projection matrix, with the choice of $(\beta_1,\beta_2,\beta_3)=(+++)$,

$$\Omega=\frac{1}{8}\left[1+\Gamma^{xy}\big(\Gamma^{12}+\Gamma^{34}+\Gamma^{56}+\Gamma^{78}\big)-\Gamma^{1234}-\Gamma^{1256}-\Gamma^{1278}\right]\mathcal{P}\,. \tag{2.23}$$

- $N=4$ SO(2)× SU(2)× SO(4) invariant projection matrix, with $(+++),(++-)$,

$$\Omega=\frac{1}{4}\left[1+\Gamma^{xy}\big(\Gamma^{12}+\Gamma^{34}\big)-\Gamma^{1234}\right]\mathcal{P}\,. \tag{2.24}$$

- $N=6$ SO(2)× SO(2)× SU(3) invariant projection matrix, with $(+++),(++-),(+-+)$,

$$\Omega=\frac{1}{8}\left[3+\Gamma^{xy}\big(3\Gamma^{12}+\Gamma^{34}+\Gamma^{56}-\Gamma^{78}\big)-\Gamma^{1234}-\Gamma^{1256}+\Gamma^{1278}\right]\mathcal{P}\,. \tag{2.25}$$

- $N=8$ SO(2)× SO(2)× SO(6) invariant projection matrix, with $(+++),(++-),(+-+),(+--)$,

$$\Omega=\frac{1}{2}(1+\Gamma^{xy12})\mathcal{P}\,. \tag{2.26}$$

- $N=8$ SO(2)× SU(4) invariant projection matrix, with $(+++),(++-),(+-+),(-++)$,

$$\Omega=\frac{1}{4}\left[2+\Gamma^{xy}\big(\Gamma^{12}+\Gamma^{34}+\Gamma^{56}-\Gamma^{78}\big)\right]\mathcal{P}\,. \tag{2.27}$$

2.4 SO(1, 1) invariant projection matrices

For SO(1, 1) invariant projection matrices, we have the following $N=1$ projection matrices:

$$\Omega=\frac{1}{16}(1+\alpha_0\Gamma^{tx})(\mathcal{P}+\alpha_1\alpha_2\mathcal{E}_1+\alpha_1\alpha_3\mathcal{E}_2+\alpha_3\mathcal{E}_3+\alpha_2\mathcal{E}_4+\alpha_1\mathcal{E}_5+\alpha_1\alpha_2\alpha_3\mathcal{E}_6+\alpha_2\alpha_3\mathcal{E}_7)\,, \tag{2.28}$$

where α_0, α_1, α_2, α_3 are four independent signs,

$$\alpha_0^2=\alpha_1^2=\alpha_2^2=\alpha_3^2=1\,. \tag{2.29}$$

Sixteen possible $N=1$ projection matrices form an orthogonal basis for the SO(1, 1) invariant projection matrices. Generic $N=k$ SO(1, 1) invariant projection matrices may be obtained straightforwardly as a k sum of the above sixteen $N=1$ projection matrices. For each sum, we may decompose

$$N=N_+ + N_-\,,\qquad N_+=n_+ + n\,,\qquad N_-=n_- + n\,, \tag{2.30}$$

such that $N_\pm$ denotes the number of $N=1$ projection matrices in the sum whose α_0 values are ±1, and n counts the number of $N=1$ projection matrix pairs which have the same α_1, α_2, α_3 values and opposite α_0 signs. There are $8!/[n_+!n_-!n!(8-n_+-n_--n)!]$ possibilities for the sum which are all equivalent to another, thanks to the SO(8) triality. Furthermore, if n is nontrivial $n\neq 0$, then the BPS configurations become SO(1, 2) invariant as $F_{\mu I}=0$ and the number of the preserved supersymmetries is automatically increased from $n_+ + n_- + 2n$ to $2(n_+ + n_- + n)$. In this case the BPS equations reduce to those of SO(1, 2) invariant BPS equations. Genuinely SO(1, 1) invariant BPS equations appear only when $n=0$. The corresponding (N_+, N_-) BPS equations are then SO(1, 1)× SO(N_+)× SO(N_-)× SO($8-N_+-N_-$) invariant with the natural restriction $N_+ + N_- \leq 8$.

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3. Classification of the BPS equations

3.1 $\mathrm{SO}(1,2)$ invariant BPS equations

The generic $N = 2$ $\mathrm{SO}(1,2)$ invariant projection matrix (2.19) leads to the following $N{=}\,2$ $\mathrm{SO}(1,2){\times}\,\mathrm{SO}(7)$ invariant BPS equations which involve three free sign factors $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1$:

$$F_{\mu I} = 0\,, \qquad\qquad \mu = t, x, y\,, \qquad I = 1, 2, \ldots, 8\,, \tag{3.1}$$

and

$$\begin{aligned}
&\alpha_1\alpha_2F_{278}{+}\alpha_2\alpha_3F_{548}{+}\alpha_3\alpha_1F_{638}{+}\alpha_1F_{234}{+}\alpha_2F_{256}{+}\alpha_3F_{357}{+}\alpha_1\alpha_2\alpha_3F_{476} \;= 0\,,\\
&\alpha_1\alpha_2F_{718}{+}\alpha_2\alpha_3F_{376}{+}\alpha_3\alpha_1F_{475}{+}\alpha_1F_{143}{+}\alpha_2F_{165}{+}\alpha_3F_{468}{+}\alpha_1\alpha_2\alpha_3F_{538} \;= 0\,,\\
&\alpha_1\alpha_2F_{456}{+}\alpha_2\alpha_3F_{267}{+}\alpha_3\alpha_1F_{168}{+}\alpha_1F_{124}{+}\alpha_2F_{478}{+}\alpha_3F_{517}{+}\alpha_1\alpha_2\alpha_3F_{258} \;= 0\,,\\
&\alpha_1\alpha_2F_{536}{+}\alpha_2\alpha_3F_{158}{+}\alpha_3\alpha_1F_{257}{+}\alpha_1F_{132}{+}\alpha_2F_{738}{+}\alpha_3F_{628}{+}\alpha_1\alpha_2\alpha_3F_{167} \;= 0\,,\\
&\alpha_1\alpha_2F_{346}{+}\alpha_2\alpha_3F_{418}{+}\alpha_3\alpha_1F_{427}{+}\alpha_1F_{678}{+}\alpha_2F_{126}{+}\alpha_3F_{137}{+}\alpha_1\alpha_2\alpha_3F_{328} \;= 0\,,\\
&\alpha_1\alpha_2F_{354}{+}\alpha_2\alpha_3F_{273}{+}\alpha_3\alpha_1F_{318}{+}\alpha_1F_{758}{+}\alpha_2F_{152}{+}\alpha_3F_{248}{+}\alpha_1\alpha_2\alpha_3F_{174} \;= 0\,,\\
&\alpha_1\alpha_2F_{128}{+}\alpha_2\alpha_3F_{236}{+}\alpha_3\alpha_1F_{245}{+}\alpha_1F_{568}{+}\alpha_2F_{348}{+}\alpha_3F_{153}{+}\alpha_1\alpha_2\alpha_3F_{146} \;= 0\,,\\
&\alpha_1\alpha_2F_{127}{+}\alpha_2\alpha_3F_{154}{+}\alpha_3\alpha_1F_{163}{+}\alpha_1F_{567}{+}\alpha_2F_{347}{+}\alpha_3F_{246}{+}\alpha_1\alpha_2\alpha_3F_{253} \;= 0\,.
\end{aligned} \tag{3.2}$$

In particular, the $\mathrm{SO}(1,2)$ invariance, the M2-brane worldvolume Lorentz symmetry, removes any worldvolume dependence, $D_\mu X_I = 0$ for all μ and I.

The above set of BPS equations can be regarded as the master equations since any $N{=}\,2k$ BPS equations can be obtained by imposing k copies of distinct $(\alpha_1, \alpha_2, \alpha_3)$ choices. The corresponding $N{=}\,2k$ BPS equations are then $\mathrm{SO}(1,2){\times}\,\mathrm{SO}(8{-}k){\times}\,\mathrm{SO}(k)$ invariant. We find for $N{=}\,14$ and $N{=}\,16$ the corresponding BPS equations are trivial, $F_{\mu I} = F_{IJK} = 0$. Other nontrivial cases are as follows.

3.1.1 $N = 2$ $\mathrm{SO}(1,2){\times}\,\mathrm{SO}(7)$ invariant BPS equations - *octonion*

With the choice of $(\alpha_1, \alpha_2, \alpha_3) = (+++)$, the $N{=}\,2$ $\mathrm{SO}(1,2){\times}\,\mathrm{SO}(7)$ invariant BPS equations (3.1), (3.3) assume a compact form:

$$F_{\mu I} = 0\,, \qquad\qquad \mathcal{C}_{IJKL}F^{JKL} = 0\,, \tag{3.3}$$

where $\mathcal{C}_{IJKL}$ is a $\mathrm{SO}(7)$ invariant four-form in eight dimensions, defined in terms of the octonionic structure constant (2.14),

$$\mathcal{C}_{ijk8} \equiv c_{ijk}\,, \qquad\quad \mathcal{C}_{ijkl} \equiv \tfrac{1}{6}\epsilon_{pqrijkl}c_{pqr} \qquad\quad \text{where} \qquad 1 \leq i, j, k, l \leq 7\,. \tag{3.4}$$

BPS states preserving $N{=}\,2k$ supersymmetries then satisfy k copies of the $N{=}\,2$ BPS equations of different α choices. The corresponding $N{=}\,2k$ BPS equations are $\mathrm{SO}(1,2){\times}\,\mathrm{SO}(k){\times}\,\mathrm{SO}(8{-}k)$ invariant, and involve k different octonionic structures.

3.1.2 $N = 4$ SO(1,2)× SO(6)× SO(2) invariant BPS equations - *complex*

The $N=4$ SO(1,2)× SO(6)× SO(2) invariant BPS equations are, with $F_{\mu I}=0$,

$$F_{IJK}\mathcal{J}^{JK} = 0\,, \qquad F_{IJK} = (1\otimes\mathcal{J}\otimes\mathcal{J} + \mathcal{J}\otimes 1\otimes\mathcal{J} + \mathcal{J}\otimes\mathcal{J}\otimes 1)_{IJK}{}^{LMN}F_{LMN}\,, \qquad (3.5)$$

where $\mathcal{J}$ is a complex structure $\mathcal{J}^2 = -1$, $\mathcal{J}^T = -\mathcal{J}$ and hence SU(4)× SO(2) invariant.

With the specific choice of α's as $(+++),(++-)$, one gets

$$\frac{1}{2}\mathcal{J}_{IJ}\Gamma^{IJ} = \Gamma^{12} + \Gamma^{34} + \Gamma^{56} + \Gamma^{78}\,. \qquad (3.6)$$

In terms of the corresponding holomorphic, anti-holomorphic coordinates $a, \bar{a} = 1,2,3,4$ and the metric $\delta^{a\bar{a}}$, the above $N=4$ SO(1,2)× SO(6)× SO(2) BPS equations (3.5) can be rewritten as

$$F_{ab}{}^{b} = F_{\bar{a}b}{}^{b} = 0\,, \qquad F_{abc} = F_{\bar{a}\bar{b}\bar{c}} = 0\,. \qquad (3.7)$$

Namely $F_{(1,2)}$, $F_{(2,1)}$ are primitive and $F_{(3,0)}= F_{(0,3)}= 0$.

We note that summing two $N=2$ projection matrices generates one complex structure. Hence in general, summing $k > 2$ of $N=2$ projection matrices will present $\binom{k}{2}$ number of complex structures to the corresponding SO(1,2)× SO(8−k)× SO(k) invariant BPS equations. The $\frac{1}{2}k(k-1)$ complex structures form singlets under SO(8−k) and are in the adjoint representation or k-dimensional two-form representation of SO(k). In fact, they correspond to the generators of SO(k). Nevertheless, the corresponding $\frac{1}{2}k(k-1)$ number of complex structures are degenerate in the sense that distinct $[\frac{k+1}{2}]$ of them are sufficient to lead to the full $N=2k$ BPS equations.

3.1.3 $N = 6$ SO(1,2)× SO(5)× SO(3) invariant BPS equations - *quarternion*

The $N=6$ SO(1,2)× SO(5)× SO(3) invariant BPS equations are, with $F_{\mu I}=0$,

$$F_{IJK}\mathcal{J}_p^{JK} = 0\,, \qquad p = 1,2,3\,, \qquad (3.8)$$

where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ are three distinct complex structures satisfying the quaternion relations:

$$\mathcal{J}_1^2 = \mathcal{J}_2^2 = \mathcal{J}_3^2 = \mathcal{J}_1\mathcal{J}_2\mathcal{J}_3 = -1\,. \qquad (3.9)$$

It is worth to note that the remaining relation of (3.5) i.e. $F_{(3,0)}= 0$ is fulfilled automatically for each complex structure.

With the specific choice of α's as $(+++),(++-),(+-+)$, one gets

$$\begin{aligned}
\frac{1}{2}\mathcal{J}_1^{IJ}\Gamma_{IJ} &= \Gamma^{12} + \Gamma^{34} + \Gamma^{56} + \Gamma^{78}\,,\\
\frac{1}{2}\mathcal{J}_2^{IJ}\Gamma_{IJ} &= \Gamma^{14} + \Gamma^{23} + \Gamma^{58} + \Gamma^{67}\,,\\
\frac{1}{2}\mathcal{J}_3^{IJ}\Gamma_{IJ} &= \Gamma^{13} + \Gamma^{42} + \Gamma^{57} + \Gamma^{86}\,.
\end{aligned} \qquad (3.10)$$

Summing three $N\!=2$ projection matrices generates one quarternion structure. Hence in general, summing $k>3$ of $N\!=2$ projection matrices will present $\binom{k}{3}$ number of quarternion structures to the corresponding $\mathrm{SO}(1,2)\times\mathrm{SO}(8-k)\times\mathrm{SO}(k)$ invariant BPS equations. The $\binom{k}{3}$ quarternion structures are singlets under $\mathrm{SO}(8-k)$ and form a k-dimensional three-form representation of $\mathrm{SO}(k)$. Nevertheless, the corresponding $\frac{1}{6}k(k-1)(k-2)$ number of quarternion structures are degenerate in the sense that distinct $[\frac{k+2}{3}]$ of them are sufficient to give the full $N\!=2k$ BPS equations.

3.1.4 $N=8$ $\mathrm{SO}(1,2)\times\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant BPS equations

The $N\!=8$ $\mathrm{SO}(1,2)\times\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant BPS equations are, with $F_{\mu I}\!=0$,

$$F_{IJK}+\frac{1}{2}F_{I}{}^{LM}\mathcal{T}_{JKLM}+\frac{1}{2}F_{J}{}^{LM}\mathcal{T}_{KILM}+\frac{1}{2}F_{K}{}^{LM}\mathcal{T}_{IJLM}=0\,, \tag{3.11}$$

where $\mathcal{T}_{IJKL}$ is a $\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant self-dual four-form. With the specific choice of α's as $(+++),(++-),(+-+),(+--)$, one gets

$$\frac{1}{4!}\mathcal{T}_{IJKL}\Gamma^{IJKL}=\Gamma^{1234}+\Gamma^{5678}\,. \tag{3.12}$$

Summing four $N\!=2$ projection matrices generates one self-dual four-form structure. Hence in general, summing $k>4$ of $N\!=2$ projection matrices will present $\binom{k}{4}$ number of self-dual four-form structures to the corresponding $\mathrm{SO}(1,2)\times\mathrm{SO}(8-k)\times\mathrm{SO}(k)$ invariant BPS equations. The $\binom{k}{4}$ self-dual four-form structures are singlets under $\mathrm{SO}(8-k)$ and form a k-dimensional four-form representation of $\mathrm{SO}(k)$. Nevertheless, the corresponding $\frac{k!}{4!(k-4)!}$ number of self-dual four-forms are degenerate in the sense that distinct $[\frac{k+3}{4}]$ of them are sufficient to give the full $N\!=2k$ BPS equations.

3.1.5 $N=10$ $\mathrm{SO}(1,2)\times\mathrm{SO}(3)\times\mathrm{SO}(5)$ invariant BPS equations

For $N\!=10$ $\mathrm{SO}(1,2)\times\mathrm{SO}(3)\times\mathrm{SO}(5)$ case there seems no novel structure to appear. One economic fashion to write the $N=10$ $\mathrm{SO}(1,2)\times\mathrm{SO}(3)\times\mathrm{SO}(5)$ invariant BPS equations is to employ a $\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant self-dual four-form and a complex structure: with $F_{\mu I}\!=0$,[2]

$$F_{IJK}+\tfrac{3}{2}F_{[I}{}^{LM}\mathcal{T}_{JK]LM}=0\,,\qquad\qquad F_{IJK}\mathcal{J}^{JK}=0\,. \tag{3.13}$$

The specific choice of α's as $(+++),(++-),(+-+),(+--),(-++)$ gives

$$\tfrac{1}{4!}\mathcal{T}_{IJKL}\Gamma^{IJKL}=\Gamma^{1234}+\Gamma^{5678}\,,\qquad\qquad \tfrac{1}{2}\mathcal{J}_{IJ}\Gamma^{IJ}=\Gamma^{18}-\Gamma^{27}+\Gamma^{36}-\Gamma^{45}\,. \tag{3.14}$$

[2] Alternatively we can express them in terms of two sets of *either* $\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant self-dual four-forms one given by (3.12) the other by $\frac{1}{2}(\Gamma_{1234}+\Gamma_{5678}+\Gamma_{1256}+\Gamma_{3478}+\Gamma_{1357}+\Gamma_{2468}+\Gamma_{1467}+\Gamma_{2358})$ *or* quarternionic complex structures one by (3.11) and the other by $\Gamma_{14}+\Gamma_{85}+\Gamma_{76}+\Gamma_{23}$, $\Gamma_{15}+\Gamma_{48}+\Gamma_{73}+\Gamma_{62}$, $\Gamma_{18}+\Gamma_{54}+\Gamma_{72}+\Gamma_{36}$.

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3.1.6 $N=12$ $\mathrm{SO}(1,2)\times\mathrm{SO}(2)\times\mathrm{SO}(6)$ invariant BPS equations

The $N=12$ $\mathrm{SO}(1,2)\times\mathrm{SO}(2)\times\mathrm{SO}(6)$ invariant BPS equations are, with $F_{\mu I}=0$,[3]

$$F_{IJK}\mathcal{T}_p^{JK}=0\,,\qquad p=1,2,3,4,5,6\,,\tag{3.15}$$

where $\mathcal{T}_p^{IJ}$'s are $\mathrm{SO}(2)\times\mathrm{SO}(6)$ covariant two-forms: fundamental under SO(6) and singlet under SO(2). With the specific choice of α's as $(+++),(++-),(+-+),(+--),(-++),(-+-)$, one gets

$$\begin{array}{ll}
\frac{1}{2}\mathcal{T}_1^{IJ}\Gamma_{IJ}=\Gamma^{14}+\Gamma^{23}\,, & \frac{1}{2}\mathcal{T}_2^{IJ}\Gamma_{IJ}=\Gamma^{67}+\Gamma^{58}\,,\\
\frac{1}{2}\mathcal{T}_3^{IJ}\Gamma_{IJ}=\Gamma^{16}+\Gamma^{25}\,, & \frac{1}{2}\mathcal{T}_4^{IJ}\Gamma_{IJ}=\Gamma^{74}+\Gamma^{83}\,,\\
\frac{1}{2}\mathcal{T}_5^{IJ}\Gamma_{IJ}=\Gamma^{17}+\Gamma^{28}\,, & \frac{1}{2}\mathcal{T}_6^{IJ}\Gamma_{IJ}=\Gamma^{35}+\Gamma^{46}\,.
\end{array}\tag{3.16}$$

3.2 SO(2) invariant BPS equations

The generic $N=2$ projection matrix (2.21) leads to the following $N=2$ $\mathrm{SO}(2)\times\mathrm{SU}(4)$ invariant BPS equations which involve three free sign factors $\beta_1^2=\beta_2^2=\beta_3^2=1$:

$$\begin{array}{llll}
F_{x1}+\beta_1F_{y2}=0\,, & F_{x3}+\beta_2F_{y4}=0\,, & F_{x5}+\beta_3F_{y6}=0\,, & F_{x7}+\beta_1\beta_2\beta_3F_{y8}=0\,,\\
F_{x2}-\beta_1F_{y1}=0\,, & F_{x4}-\beta_2F_{y3}=0\,, & F_{x6}-\beta_3F_{y5}=0\,, & F_{x8}-\beta_1\beta_2\beta_3F_{y7}=0\,,
\end{array}\tag{3.17}$$

and

$$\begin{array}{rl}
F_{t1}+\beta_2F_{134}+\beta_3F_{156}+\beta_1\beta_2\beta_3F_{178}=0\,, & F_{135}-\beta_1\beta_2F_{245}-\beta_2\beta_3F_{146}-\beta_3\beta_1F_{236}=0\,,\\
F_{t2}+\beta_2F_{234}+\beta_3F_{256}+\beta_1\beta_2\beta_3F_{278}=0\,, & F_{136}-\beta_1\beta_2F_{246}+\beta_2\beta_3F_{145}+\beta_3\beta_1F_{235}=0\,,\\
F_{t3}+\beta_1F_{312}+\beta_3F_{356}+\beta_1\beta_2\beta_3F_{378}=0\,, & F_{137}-\beta_1\beta_2F_{247}-\beta_2\beta_3F_{238}-\beta_3\beta_1F_{148}=0\,,\\
F_{t4}+\beta_1F_{412}+\beta_3F_{456}+\beta_1\beta_2\beta_3F_{478}=0\,, & F_{138}-\beta_1\beta_2F_{248}+\beta_2\beta_3F_{237}+\beta_3\beta_1F_{147}=0\,,\\
F_{t5}+\beta_1F_{512}+\beta_2F_{534}+\beta_1\beta_2\beta_3F_{578}=0\,, & F_{157}-\beta_1\beta_2F_{168}-\beta_2\beta_3F_{258}-\beta_3\beta_1F_{267}=0\,,\\
F_{t6}+\beta_1F_{612}+\beta_2F_{634}+\beta_1\beta_2\beta_3F_{678}=0\,, & F_{158}+\beta_1\beta_2F_{167}+\beta_2\beta_3F_{257}-\beta_3\beta_1F_{268}=0\,,\\
F_{t7}+\beta_1F_{712}+\beta_2F_{734}+\beta_3F_{756}=0\,, & F_{357}-\beta_1\beta_2F_{368}-\beta_2\beta_3F_{467}-\beta_3\beta_1F_{458}=0\,,\\
F_{t8}+\beta_1F_{812}+\beta_2F_{834}+\beta_3F_{856}=0\,, & F_{358}+\beta_1\beta_2F_{367}-\beta_2\beta_3F_{468}+\beta_3\beta_1F_{457}=0\,.
\end{array}\tag{3.18}$$

The above set of BPS equations can be regarded as the master equations since any $N=2k$ SO(2)[5] invariant BPS equations corresponding to the projection matrices (2.23 - 2.27) can be obtained by imposing k copies of distinct $(\beta_1,\beta_2,\beta_3)$ choices. We find, among them, the $N=8$ $\mathrm{SO}(2)\times\mathrm{SU}(4)$ invariant projection matrix (2.27) leads to the trivial BPS configuration $F_{\mu I}=F_{IJK}=0$. Other nontrivial cases are as follows.

[3]Of course, the above $N=12$ BPS equations can be obtained by imposing a pair of two distinct quarternionic BPS equations (3.8). There are $\frac{1}{2}\binom{6}{3}=10$ such pairs and any of them leads to the same $N=12$ BPS equations. For example we may choose one quarternion structure from (3.11) and the other by $\Gamma^{12}+\Gamma^{87}+\Gamma^{56}+\Gamma^{43}$, $\Gamma^{17}+\Gamma^{28}+\Gamma^{53}+\Gamma^{64}$, $\Gamma^{18}+\Gamma^{72}+\Gamma^{54}+\Gamma^{36}$, corresponding to the α choices $(+++),(++-),(+-+)$ and $(+--),(-++),(-+-)$.

3.2.1 $N = 2$ SO(2)× SU(4) invariant BPS equations

The $N = 2$ SO(2)× SU(4) invariant BPS equations corresponding to the projection matrix (2.23) or the choice $(\beta_1, \beta_2, \beta_3) = (+++)$ in (3.17) and (3.18) assume a compact form, up to Hermitian conjugation:

$$F_{z\bar{a}} = 0\,, \qquad F_{ta} - iF_{ab}{}^{b} = 0\,, \qquad F_{abc} = 0\,, \tag{3.19}$$

provided we complexify the SO(8) coordinates by the complex structure $\Gamma_{12}+\Gamma_{34}+\Gamma_{56}+\Gamma_{78}$, to introduce the holomorphic and anti-holomorphic variables $a, \bar{a} = 1, 2, 3, 4$ such that the metric is $\delta_{a\bar{a}}$ and

$$\begin{aligned} D_z &= \frac{1}{\sqrt{2}}(D_x - iD_y)\,, & D_{\bar{z}} &= \frac{1}{\sqrt{2}}(D_x + iD_y)\,, \\ F_{za} &= \frac{1}{\sqrt{2}}(D_z X_{2a-1} - iD_z X_{2a})\,, & F_{z\bar{a}} &= \frac{1}{\sqrt{2}}(D_z X_{2\bar{a}-1} + iD_z X_{2\bar{a}})\,. \end{aligned} \tag{3.20}$$

3.2.2 $N = 4$ SO(2)× SU(2)× SO(4) invariant BPS equations

The $N = 4$ SO(2)× SU(2)× SO(4) invariant BPS equations corresponding to the projection matrix (2.24) are, up to Hermitian conjugation,

$$F_{z\bar{a}} = 0\,, \quad F_{zp} = 0\,, \quad F_{pab} = 0\,, \quad F_{tI} - iF_{Ia}{}^{a} = 0\,, \quad F_{Ipq} + \frac{1}{2}\epsilon_{pqrs}F_{I}{}^{rs} = 0\,, \tag{3.21}$$

where $I = 1, 2, \ldots, 8$, $p, q, r, s = 5, 6, 7, 8$, ϵ_{pqrs} is a totally anti-symmetric tensor with $\epsilon_{5678} = 1$ and $a, b, \bar{a} = 1, 2$ such that the SO(4) ⊂ SO(8) coordinates are complexified by the complex structure $\Gamma_{12}+\Gamma_{34}$.

3.2.3 $N = 6$ SO(2)× SO(2)× SU(3) invariant BPS equations

The $N = 6$ SO(2)× SO(2)× SU(3) invariant BPS equations corresponding to the projection matrix (2.25) are, up to Hermitian conjugation,

$$\begin{aligned} &F_{z\bar{\omega}} = 0\,, && F_{za} = 0\,, && F_{z\bar{a}} = 0\,, && F_{t\omega} - i\frac{1}{3}F_{\omega a}{}^{a} = 0\,, \\ &F_{ta} - iF_{a\omega\bar{\omega}} = 0\,, && F_{\omega ab} = 0\,, && F_{ab\bar{c}} = 0\,, && F_{\omega a\bar{b}} - \frac{1}{3}(F_{\omega c}{}^{c})\delta_{a\bar{b}} = 0\,, \end{aligned} \tag{3.22}$$

where $a, \bar{a} = 1, 2, 3$ such that we complexify the SO(6) ⊂ SO(8) coordinates by the complex structure $\Gamma_{34}+\Gamma_{56}+\Gamma_{87}$ and also set separately for SO(2) ⊂ SO(8),

$$F_{z\omega} \equiv \tfrac{1}{\sqrt{2}}(F_{z1} - iF_{z2})\,, \qquad F_{z\bar{\omega}} \equiv \tfrac{1}{\sqrt{2}}(F_{z1} + iF_{z2})\,. \tag{3.23}$$

3.2.4 $N = 8$ SO(2)× SO(2)× SO(6) invariant BPS equations

The $N = 8$ SO(2)× SO(2)× SO(6) invariant BPS equations corresponding to the projection matrix (2.26) are, up to Hermitian conjugation,

$$F_{z\bar{\omega}} = 0\,, \qquad F_{zp} = 0\,, \qquad F_{tI} - iF_{I\omega\bar{\omega}} = 0\,, \qquad F_{Ipq} = 0\,, \tag{3.24}$$

where $I = 1, 2, \ldots, 8$, $p = 3, 4, 5, 6, 7, 8$ and we complexify the SO(2) ⊂ SO(8) coordinates by the complex structure Γ_{12} to employ (3.23).

3.3 SO$(1,1)$ invariant BPS equations

The generic $N=1$ projection matrix (2.28) leads to the following $N=1$ SO$(1,1)\times$SO(7) invariant BPS equations which involve four free signs $\alpha_0^2=\alpha_1^2=\alpha_2^2=\alpha_3^2=1$:

$$\begin{array}{l}
F_{tI}-\alpha_0 F_{xI}=0\,, \qquad\qquad I=1,2,\ldots,8\,,\\
\alpha_0F_{y1}-\alpha_1\alpha_2F_{278}-\alpha_2\alpha_3F_{548}-\alpha_3\alpha_1F_{638}-\alpha_1F_{234}-\alpha_2F_{256}-\alpha_3F_{357}-\alpha_1\alpha_2\alpha_3F_{476}=0\,,\\
\alpha_0F_{y2}-\alpha_1\alpha_2F_{718}-\alpha_2\alpha_3F_{376}-\alpha_3\alpha_1F_{475}-\alpha_1F_{143}-\alpha_2F_{165}-\alpha_3F_{468}-\alpha_1\alpha_2\alpha_3F_{538}=0\,,\\
\alpha_0F_{y3}-\alpha_1\alpha_2F_{456}-\alpha_2\alpha_3F_{267}-\alpha_3\alpha_1F_{168}-\alpha_1F_{124}-\alpha_2F_{478}-\alpha_3F_{517}-\alpha_1\alpha_2\alpha_3F_{258}=0\,,\\
\alpha_0F_{y4}-\alpha_1\alpha_2F_{536}-\alpha_2\alpha_3F_{158}-\alpha_3\alpha_1F_{257}-\alpha_1F_{132}-\alpha_2F_{738}-\alpha_3F_{628}-\alpha_1\alpha_2\alpha_3F_{167}=0\,,\\
\alpha_0F_{y5}-\alpha_1\alpha_2F_{346}-\alpha_2\alpha_3F_{418}-\alpha_3\alpha_1F_{427}-\alpha_1F_{678}-\alpha_2F_{126}-\alpha_3F_{137}-\alpha_1\alpha_2\alpha_3F_{328}=0\,,\\
\alpha_0F_{y6}-\alpha_1\alpha_2F_{354}-\alpha_2\alpha_3F_{273}-\alpha_3\alpha_1F_{318}-\alpha_1F_{758}-\alpha_2F_{152}-\alpha_3F_{248}-\alpha_1\alpha_2\alpha_3F_{174}=0\,,\\
\alpha_0F_{y7}-\alpha_1\alpha_2F_{128}-\alpha_2\alpha_3F_{236}-\alpha_3\alpha_1F_{245}-\alpha_1F_{568}-\alpha_2F_{348}-\alpha_3F_{153}-\alpha_1\alpha_2\alpha_3F_{146}=0\,,\\
\alpha_0F_{y8}+\alpha_1\alpha_2F_{127}+\alpha_2\alpha_3F_{154}+\alpha_3\alpha_1F_{163}+\alpha_1F_{567}+\alpha_2F_{347}+\alpha_3F_{246}+\alpha_1\alpha_2\alpha_3F_{253}=0\,.
\end{array} \tag{3.25}$$

The above set of BPS equations can be regarded as the master equations for generic SO$(1,1)$ invariant BPS equations. One can classify the BPS equations according to the decomposition of the number of preserved supersymmetries as (N_+,N_-) (2.30). Among others, below we spell explicitly $(N_+,0)$ as well as (N,N) BPS equations with $N_+=1,2,\ldots,7,\;\; N=1,2,3,4$.

3.3.1 $(N_+,N_-)=(1,0)$ SO$(1,1)\times$SO(7) invariant BPS equations - *octonion*

With the choice of $(\alpha_0,\alpha_1,\alpha_2,\alpha_3)=(++++)$, the $(N_+,N_-)=(1,0)$ SO$(1,1)\times$SO(7) invariant BPS equations (3.26) assume a compact form:

$$F_{tI}-F_{xI}=0\,, \qquad\qquad F_{yI}-\tfrac{1}{6}\mathcal{C}_{IJKL}F^{JKL}=0\,, \tag{3.26}$$

which generalizes the $N=2$ SO$(1,2)\times$SO(7) invariant BPS equations (3.3).

3.3.2 $(N_+,N_-)=(2,0)$ SO$(1,1)\times$SO$(2)\times$SO(6) invariant BPS equations - *complex*

The $(N_+,N_-)=(2,0)$ SO$(1,1)\times$SO$(2)\times$SO(6) invariant BPS equations are, with $F_{tI}-F_{xI}=0$,

$$\mathcal{J}^{IJ}F_{yJ}+\frac{1}{2}F^{I}{}_{JK}\mathcal{J}^{JK}=0\,,\quad F_{IJK}=(1\otimes\mathcal{J}\otimes\mathcal{J}+\mathcal{J}\otimes1\otimes\mathcal{J}+\mathcal{J}\otimes\mathcal{J}\otimes1)_{IJK}{}^{LMN}F_{LMN}\,, \tag{3.27}$$

which generalizes the $N=4$ SO$(1,2)\times$SO$(6)\times$SO(2) invariant BPS equations (3.5).

3.3.3 $(N_+,N_-)=(3,0)$ SO$(1,1)\times$SO$(3)\times$SO(5) invariant BPS equations - *quaternion*

The $(N_+,N_-)=(3,0)$ SO$(1,1)\times$SO$(3)\times$SO(5) invariant BPS equations are, with $F_{tI}-F_{xI}=0$,

$$\mathcal{J}_p^{IJ}F_{yJ}+\frac{1}{2}F^{I}{}_{JK}\mathcal{J}_p^{JK}=0\,, \qquad p=1,2,3\,, \tag{3.28}$$

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where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ are three distinct complex structures satisfying the quaternion relations, $\mathcal{J}_1^2 = \mathcal{J}_2^2 = \mathcal{J}_3^2 = \mathcal{J}_1\mathcal{J}_2\mathcal{J}_3 = -1$ (3.11). It is worth to note that the remaining relation of (3.27) $F_{(3,0)} = 0$ is fulfilled automatically for each complex structure. Eq. (3.28) generalizes the $N = 6$ SO(1, 2)× SO(5)× SO(3) invariant BPS equations (3.8).

3.3.4 $(N_+, N_-) = (4, 0)$ SO(1, 1)× SO(4)× SO(4) invariant BPS equations

The $(N_+, N_-) = (4, 0)$ SO(1, 1)× SO(4)× SO(4) invariant BPS equations are, with $F_{tI} - F_{xI} = 0$,

$$\mathcal{T}_{IJKL} F_y{}^L + F_{IJK} + \frac{1}{2} F_I{}^{LM} \mathcal{T}_{JKLM} + \frac{1}{2} F_J{}^{LM} \mathcal{T}_{KILM} + \frac{1}{2} F_K{}^{LM} \mathcal{T}_{IJLM} = 0 \,, \qquad (3.29)$$

where $\mathcal{T}_{IJKL}$ is a SO(4)× SO(4) invariant self-dual four-form (3.12). Eq. (3.29) generalizes the $N = 8$ SO(1, 2)× SO(4)× SO(4) invariant BPS equations (3.11). Some mass deformations of the above BPS equations are studied in ref. [23].

3.3.5 $(N_+, N_-) = (5, 0)$ SO(1, 1)× SO(5)× SO(3) invariant BPS equations

The $(N_+, N_-) = (5, 0)$ SO(1, 1)× SO(5)× SO(3) invariant BPS equations are, with $F_{tI} - F_{xI} = 0$,

$$\mathcal{T}_{IJKL} F_y{}^L + F_{IJK} + \tfrac{3}{2} F_{[I}{}^{LM} \mathcal{T}_{JK]LM} = 0 \,, \qquad \mathcal{J}^{IJ} F_{yJ} + \tfrac{1}{2} F_{IJK} \mathcal{J}^{JK} = 0 \,, \qquad (3.30)$$

where $\mathcal{T}_{IJKL}$ and $\mathcal{J}^{IJ}$ are given in (3.14). Eq. (3.30) generalizes the $N = 10$ SO(1, 2)× SO(3)× SO(5) invariant BPS equations (3.13).

3.3.6 $(N_+, N_-) = (6, 0)$ SO(1, 1)× SO(6)× SO(2) invariant BPS equations

The $(N_+, N_-) = (6, 0)$ SO(1, 1)× SO(6)× SO(2) invariant BPS equations are, $F_{tI} - F_{xI} = 0$,

$$\mathcal{T}_p^{IJ} F_{yJ} + \frac{1}{2} F^I{}_{JK} \mathcal{T}_p^{JK} = 0 \,, \qquad p = 1, 2, 3, 4, 5, 6 \,, \qquad (3.31)$$

where six of two-forms $\mathcal{T}_p$, $p = 1, 2, \ldots, 6$ are given in (3.16). Eq. (3.31) generalizes the $N = 12$ SO(1, 2)× SO(2)× SO(6) invariant BPS equations (3.15).

3.3.7 $(N_+, N_-) = (7, 0)$ SO(1, 1)× SO(7) invariant BPS equations

The $(N_+, N_-) = (7, 0)$ SO(1, 1)× SO(7) invariant BPS equations are, with $F_{tI} - F_{xI} = 0$,

$$\mathcal{T}_p^{IJ} F_{yJ} + \frac{1}{2} F^I{}_{JK} \mathcal{T}_p^{JK} = 0 \,, \qquad p = 1, 2, 3, 4, 5, 6, 7 \,. \qquad (3.32)$$

Here we have seven of two-forms, six given by (3.16) and last one by

$$\frac{1}{2} \mathcal{T}_7^{IJ} \Gamma_{IJ} = \Gamma^{13} + \Gamma^{57} \,. \qquad (3.33)$$

They form a fundamental representation of SO(7).

3.3.8 $(N_+,N_-)=(1,1)$ $\mathrm{SO}(1,1)\times\mathrm{SO}(6)$ invariant BPS equations

The $(N_+,N_-)=(1,1)$ $\mathrm{SO}(1,1)\times\mathrm{SO}(6)$ invariant BPS equations are, with $F_{tI}=F_{xI}=0$, best expressed in complex coordinates,

$$F_{ab}{}^{b}=0\,, \qquad\qquad F_{y\bar{a}}-\tfrac{1}{3}\epsilon_{\bar{a}}{}^{bcd}F_{bcd}=0\,. \tag{3.34}$$

3.3.9 $(N_+,N_-)=(2,2)$ $\mathrm{SO}(1,1)\times\mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{SO}(4)$ invariant BPS equations

The $(N_+,N_-)=(2,2)$ $\mathrm{SO}(1,1)\times\mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{SO}(4)$ invariant BPS equations are, with $F_{tI}=F_{xI}=0$,

$$(3\mathcal{J}^{[IJ}\mathcal{J}^{K]L}-\mathcal{T}^{IJKL})F_{yL}+F^{IJK}+\frac{3}{2}F^{[I}{}_{LM}\mathcal{T}^{JK]LM}=0\,, \tag{3.35}$$

where $\mathcal{J}^{IJ}$ is the complex structure of $\Gamma^{12}+\Gamma^{34}+\Gamma^{56}+\Gamma^{78}$ (3.6) and $\mathcal{T}^{IJKL}$ is the self-dual $\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant four-form tensor of $\Gamma^{1234}+\Gamma^{5678}$ (3.12).

3.3.10 $(N_+,N_-)=(3,3)$ $\mathrm{SO}(1,1)\times\mathrm{SO}(3)\times\mathrm{SO}(3)\times\mathrm{SO}(2)$ invariant BPS equations

We present the $(N_+,N_-)=(3,3)$ $\mathrm{SO}(1,1)\times\mathrm{SO}(3)\times\mathrm{SO}(3)\times\mathrm{SO}(2)$ invariant BPS equations with a pair of quarternion structures, one from (3.11) and the other from $\Gamma^{12}+\Gamma^{87}+\Gamma^{56}+\Gamma^{43}$, $\Gamma^{17}+\Gamma^{28}+\Gamma^{53}+\Gamma^{64}$, $\Gamma^{18}+\Gamma^{72}+\Gamma^{54}+\Gamma^{36}$. With $F_{tI}=F_{xI}=0$ they are

$$\mathcal{J}_p^{IJ}F_{yJ}+\frac{1}{2}F^{I}{}_{JK}\mathcal{J}_p^{JK}=0\,, \qquad \hat{\mathcal{J}}_p^{IJ}F_{yJ}-\frac{1}{2}F^{I}{}_{JK}\hat{\mathcal{J}}_p^{JK}=0\,, \qquad p=1,2,3\,. \tag{3.36}$$

3.3.11 $(N_+,N_-)=(4,4)$ $\mathrm{SO}(1,1)\times\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant BPS equations

The $(N_+,N_-)=(4,4)$ $\mathrm{SO}(1,1)\times\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant BPS equations are, with $F_{tI}=F_{xI}=0$, in terms of the self-dual $\times\,\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant four-form tensor,

$$\mathcal{T}^{IJKL}F_{yL}+F^{IJK}=0\,. \tag{3.37}$$

Especially among all the half BPS cases i.e. $N_++N_-=8$, only the case $(N_+,N_-)=(4,4)$ leads to the nontrivial BPS equations.

4. Discussion

In this paper we studied and identified a number of BPS equations for the multiple M2-brane theory proposed recently by Bagger and Lambert. We employed a method which had been successfully applied to several analogous problems. One first constructs the basic projection matrices for the supersymmetry parameters, and then obtain the corresponding BPS equations. Our classifications are complete for $\mathrm{SO}(1,2)$ as well as $\mathrm{SO}(2)^{\mathbf{5}}$ invariant BPS equations, while may be not for $\mathrm{SO}(1,1)$ invariant cases.

The BPS equations with different types and numbers of preserved supersymmetries are derived in terms of the associated tensors which are invariant under the symmetry group of the relevant BPS equations. In particular we derived three types of half BPS equations, which we recall:

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- $N=8$ SO(1,2)× SO(4)× SO(4) invariant BPS equations (3.11)

$$F_{\mu I}=0\,,\qquad F_{IJK}+\frac{1}{2}F_I{}^{LM}\mathcal{T}_{JKLM}+\frac{1}{2}F_J{}^{LM}\mathcal{T}_{KILM}+\frac{1}{2}F_K{}^{LM}\mathcal{T}_{IJLM}=0\,. \quad (4.1)$$

- $N=8$ SO(2)× SO(2)× SO(6) invariant BPS equations (3.24)

$$F_{z\bar{\omega}}=0\,,\qquad F_{zp}=0\,,\qquad F_{tI}-iF_{I\omega\bar{\omega}}=0\,,\qquad F_{Ipq}=0\,, \quad (4.2)$$

  where $I=1,2,\ldots,8$, $p=3,4,5,6,7,8$, and $\omega,\bar{\omega}$ are complex coordinates for SO(2) ⊂ SO(8).

- $(N_+,N_-)=(4,4)$ SO(1,1)× SO(4)× SO(4) invariant BPS equations (3.37)

$$F_{tI}=F_{xI}=0\,,\qquad \mathcal{T}^{IJKL}F_{yL}+F^{IJK}=0\,. \quad (4.3)$$

The BPS equations for different number of supersymmetries exhibit the division algebra structures: octonion, quarternion or complex. Let us take the Lorentz invariant type as examples. For the least supersymmetric configurations preserving 1/8 supersymmetries, the relevant symmetry is SO(1,2)× SO(7) and the BPS equations can be elegantly written in terms of the invariant four-form which has close relation to octonions. For 1/4-BPS equations the symmetry is SO(1,2)× SO(6)× SO(2) and a complex structure appears. We next have 3/8 SO(1,2)× SO(5)× SO(3) invariant BPS equations, which are naturally best expressed in terms of quarternions or hyper-Kähler structure. In addition, for 1/2-BPS equations we have the SO(4) × SO(4) invariant self-dual four-form structure. We have also identified the exotic classes with more than 1/2 supersymmetry. Apparently the governing symmetries include more than one hyper-Kähler structures, but we have not been able to express the BPS equations in a succinct way. The true mathematical identity of such systems certainly deserves more careful study.

The explicit solutions of the BPS equations will give the spectrum of supersymmetric solitons in Bagger-Lambert theory. It is natural to ask the $\mathcal{M}$-theory interpretation of such objects. The real scalar fields X^I describe the locations of M2-branes in the transverse $\mathbb{R}^8$. The spatial dependence of X^I thus informs us on the shape of M2-branes, or how they are embedded in the transverse $\mathbb{R}^8$. Eq. (3.17) and the subsequent analysis clearly suggest that the M2-brane worldvolume should occupy holomorphic curves, which is natural for supersymmetry. Likewise, time-dependence of the scalar field obviously implies that there is momentum along the particular direction. The three-algebra terms F_{IJK} describe the truly $\mathcal{M}$-theoretic phenomena: polarization of multiple M2-branes into M5-branes. Generically the BPS equations are given as various combinations of such basic building blocks, and more detailed descriptions with explicit solutions will be reported in a separate publication.

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A. Gamma matrices and octonions

The eleven-dimensional 32×32 gamma matrices Γ^M, $M=\mu,I$, $\mu=t,x,y$, $I=1,2,\ldots,8$ in the Bagger-Lambert theory naturally decompose into two parts: SO(1, 2) the M2-brane worldvolume and SO(8) the transverse space,

$$\Gamma^t=\epsilon\otimes\gamma_{(9)}\,,\quad \Gamma^x=\sigma_1\otimes\gamma_{(9)}\,,\quad \Gamma^y=\sigma_3\otimes\gamma_{(9)}\,,\quad \Gamma^I=1\otimes\gamma^I\,,\qquad I=1,2,\ldots,8\,. \tag{A.1}$$

Here γ^I's are the 16×16 gamma matrices in the eight-dimensional Euclidean space and $\gamma_{(9)}\equiv\gamma_{12\cdots8}$. Clearly the SO(1, 2) projection constraint (2.4) coincides with that of SO(8),

$$\Gamma^{txy}=1\otimes\gamma_{(9)}\,. \tag{A.2}$$

This is consistent with the fact that the product of all the eleven-dimensional gamma matrices leads to the identity $\Gamma^{txy123\cdots8}=1$.

Now we recall the seven quantities $\mathcal{E}_i$, $i=1,2,3\cdots,7$ (2.13). In the above choice of gamma matrices we have

$$\mathcal{E}_i=1\otimes E_i\,,\qquad \mathcal{P}=1\otimes P\,, \tag{A.3}$$

where as in (2.13)

$$\begin{aligned} &E_1=\gamma_{8127}P\,, && E_2=\gamma_{8163}P\,, && E_3=\gamma_{8246}P\,, && E_4=\gamma_{8347}P\,,\\ &E_5=\gamma_{8567}P\,, && E_6=\gamma_{8253}P\,, && E_7=\gamma_{8154}P\,, && P=\frac{1}{2}(1+\gamma_{(9)})\,. \end{aligned} \tag{A.4}$$

The subscript spatial indices of the gamma matrices are organized such that the three indices after the common 8 are identical to those of the totally anti-symmetric octonionic structure constants (2.14). It is straightforward to see that E_i forms a representation of the "square" of the octonions on the eight-dimensional chiral space,

$$E_iE_j=\delta_{ij}P+c^2_{ijk}\,E_k\,,\qquad E_i\equiv e_i\otimes e_i\,. \tag{A.5}$$

Since they commute each other, they form a maximal set of the mutually commuting traceless symmetric and real matrices of the definite chirality $\gamma_{(9)}E_i=E_i$. In fact, one can construct a SO(8) symmetric and real gamma matrix representation which makes all E_i's be simultaneously diagonal, utilizing the octonionic structure constants:

$$\gamma_I=\begin{pmatrix}0&\rho_I\\(\rho_I)^T&0\end{pmatrix},\qquad \rho_I(\rho_J)^T+\rho_J(\rho_I)^T=2\delta_{IJ}\,,\qquad \gamma_{(9)}=\gamma_{12345678}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{A.6}$$

Here ρ_I, $I=1,2,\ldots,8$ are 8×8 real matrices given by[4]

$$\rho_i=-(\rho_i)^T=\begin{pmatrix}c_i&-n_i\\(n_i)^T&0\end{pmatrix},\qquad i=1,2,\ldots,7\,,\qquad \rho_8=1\,, \tag{A.7}$$

[4]In particular, ρ_i, $1\le i\le 7$ correspond to the Majorana gamma matrices in Euclidean seven dimensions $\rho_i\rho_j+\rho_j\rho_i=-2\delta_{ij}$.

and c_i is a 7×7 real matrix whose j, k component is nothing but the octonionic structure constant c_{ijk} (2.14), while n_i is a seven-dimensional unit vector of which the jth component is defined to be $\delta_i{}^j$.

In the above choice of Majorana gamma matrix representation, all the E_i's and P are diagonal,

$$\begin{aligned}
E_1 &= \mathrm{diag}(+1,+1,-1,-1,-1,-1,+1,+1,0,0,0,0,0,0,0,0)\,,\\
E_2 &= \mathrm{diag}(+1,-1,+1,-1,-1,+1,-1,+1,0,0,0,0,0,0,0,0)\,,\\
E_3 &= \mathrm{diag}(-1,+1,-1,+1,-1,+1,-1,+1,0,0,0,0,0,0,0,0)\,,\\
E_4 &= \mathrm{diag}(-1,-1,+1,+1,-1,-1,+1,+1,0,0,0,0,0,0,0,0)\,,\\
E_5 &= \mathrm{diag}(-1,-1,-1,-1,+1,+1,+1,+1,0,0,0,0,0,0,0,0)\,,\\
E_6 &= \mathrm{diag}(-1,+1,+1,-1,+1,-1,-1,+1,0,0,0,0,0,0,0,0)\,,\\
E_7 &= \mathrm{diag}(+1,-1,-1,+1,+1,-1,-1,+1,0,0,0,0,0,0,0,0)\,,\\
P &= \mathrm{diag}(+1,+1,+1,+1,+1,+1,+1,+1,0,0,0,0,0,0,0,0)\,,
\end{aligned} \tag{A.8}$$

and the SO(8) triality among $\mathbf{8}_\mathrm{v}$, $\mathbf{8}_+$, $\mathbf{8}_-$ is apparent as the $\mathbf{8}_\mathrm{v}$ generators decompose into the $\mathbf{8}_+$ and $\mathbf{8}_-$ generators,

$$\gamma_{IJ} = \begin{pmatrix} \rho_{[I}\rho^T_{J]} & 0 \\ 0 & \rho^T_{[I}\rho_{J]} \end{pmatrix} . \tag{A.9}$$

With the identity $e_8 \equiv 1$, the octonion algebra now spells completely:

$$e_I e_J = (\rho_I)_{JK}\, e_K\,, \qquad\qquad I, J, K = 1, 2, \ldots, 8\,. \tag{A.10}$$

Finally let us consider a self-dual four-form and contract it with the SO(8) gamma matrices Γ^{IJKL}, such as $\Upsilon_4\mathcal{P}$ in (2.10). Clearly utilizing the SO(8) triality, one can diagonalize $\Upsilon_4\mathcal{P}$ to express it as a linear combination of $\mathcal{E}_i$'s. This shows that the canonical form of a self-dual four-form in eight dimensions indeed takes the form (2.12): namely the non-vanishing independent components are only those seven which are contracted to $\mathcal{E}_i$'s.

B. SO(2) invariant projection matrix

Here we derive the most general form of the 32×32 projection matrices Ω which are invariant under the Cartan subalgebra $\mathrm{SO}(2)^{\mathbf{5}}$ of SO(10), satisfying in addition to the conditions (2.8),

$$[\Gamma^{xy}, \Omega] = 0\,, \qquad [\Gamma^{12}, \Omega] = 0\,, \qquad [\Gamma^{34}, \Omega] = 0\,, \qquad [\Gamma^{56}, \Omega] = 0\,, \qquad [\Gamma^{78}, \Omega] = 0\,. \tag{B.1}$$

As (2.17), they assume the general form:

$$\Omega = \left[c + \Gamma^{xy}\left(a_1\Gamma^{12} + a_2\Gamma^{34} + a_3\Gamma^{56} + a_4\Gamma^{78}\right) + b_1\Gamma^{1234} + b_2\Gamma^{1256} + b_3\Gamma^{1278}\right]\mathcal{P}\,, \tag{B.2}$$

where $c, a_1, \ldots, b_3$ are eight *a priori* unknown real constants which must be determined by requiring the remaining condition $\Omega^2 = \Omega$. In particular the number of the preserved supersymmetries is related to the constant c by

$$N = \mathrm{Tr}\Omega = 16c\,. \tag{B.3}$$

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It is convenient to reparameterize the four constants a_1, a_2, a_3, a_4 by four other constants e_1, e_2, e_3, e_4

$$
\begin{aligned}
e_1 &= 2(a_1 + a_2 + a_3 + a_4)\,, & e_2 &= 2(a_1 + a_2 - a_3 - a_4)\,, \\
e_3 &= 2(a_1 - a_2 + a_3 - a_4)\,, & e_4 &= 2(-a_1 + a_2 + a_3 - a_4)\,,
\end{aligned} \tag{B.4}
$$

and the other four constants c, b_1, b_2, b_3 by another set of four constants f_1, f_2, f_3, f_4

$$
\begin{aligned}
f_1 &= 2c - 1 - 2b_1 - 2b_2 - 2b_3\,, & f_2 &= 2c - 1 - 2b_1 + 2b_2 + 2b_3\,, \\
f_3 &= 2c - 1 + 2b_1 - 2b_2 + 2b_3\,, & f_4 &= 2c - 1 + 2b_1 + 2b_2 - 2b_3\,.
\end{aligned} \tag{B.5}
$$

It follows that

$$
\begin{aligned}
a_1 &= \frac{1}{8}(e_1 + e_2 + e_3 - e_4)\,, & a_2 &= \frac{1}{8}(e_1 + e_2 - e_3 + e_4)\,, \\
a_3 &= \frac{1}{8}(e_1 - e_2 + e_3 + e_4)\,, & a_4 &= \frac{1}{8}(e_1 - e_2 - e_3 - e_4)\,, \\
b_1 &= \frac{1}{8}(-f_1 - f_2 + f_3 + f_4)\,, & b_2 &= \frac{1}{8}(-f_1 + f_2 - f_3 + f_4)\,, \\
b_3 &= \frac{1}{8}(-f_1 + f_2 + f_3 - f_4)\,, & c &= \frac{1}{8}(f_1 + f_2 + f_3 + f_4 + 4)\,.
\end{aligned} \tag{B.6}
$$

Straightforward calculation shows that $\Omega^2 = \Omega$ is equivalent for each $a = 1, 2, 3, 4$ to

$$
f_a e_a = 0\,, \qquad e_a^2 = (1 + f_a)(1 - f_a) \qquad \text{not } a \text{ sum}\,. \tag{B.7}
$$

Hence for each a we have four possible solutions:

$$
e_a = 0\,,\ f_a = +1\,; \quad e_a = 0\,,\ f_a = -1\,; \quad e_a = +1\,,\ f_a = 0\,; \quad e_a = -1\,,\ f_a = 0\,. \tag{B.8}
$$

Consequently from (B.3) and (B.6), the possible values of c are 0, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, $\frac{5}{8}$, $\frac{6}{8}$, $\frac{7}{8}$, 1, so that the number of the preserved supersymmetries N is an even number between zero and sixteen. The basic building blocks of all the possible projection matrices are those of $N = 2$ given by

$$
\begin{aligned}
\Omega &= \frac{1}{8}\left[1 + \Gamma^{xy}\left(\beta_1\Gamma^{12} + \beta_2\Gamma^{34} + \beta_3\Gamma^{56} + \beta_1\beta_2\beta_3\Gamma^{78}\right) - \beta_1\beta_2\Gamma^{1234} - \beta_3\beta_1\Gamma^{1256} - \beta_2\beta_3\Gamma^{1278}\right]\mathcal{P} \\
&= \frac{1}{8}(1 + \beta_1\Gamma^{xy12})(1 + \beta_2\Gamma^{xy34})(1 + \beta_3\Gamma^{xy56})\mathcal{P}\,,
\end{aligned} \tag{B.9}
$$

where β_1, β_2, β_3 are three independent signs,

$$
\beta_1^2 = \beta_2^2 = \beta_3^2 = 1\,. \tag{B.10}
$$

There are eight possible $N = 2$ projection matrices which are orthogonal to each other. By summing k of them, all the other generic projection matrices preserving $N = 2k$ supersymmetries can be obtained.

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